W-symmetries on the Homogeneous Space $G/U(1)^r$

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Abstract

A construction of W-symmetries is given only in terms of the nonlocal fields (parafermions ψ_α), which take values on the homogeneous space $G/U(1)^r$, where G is a simply connected compact Lie group manifold (its accompanying Lie algebra $\mathcal{G}$ is a simple one of rank r). Only certain restriction of the root set of Lie algebra on which the parafermionic fields take values are satisfied, then a consistent and non-trivial extension of the stress momentum tensor may exist. For arbitrary simple-laced algebras, i.e. the $A - D - E$ cases, a more detailed discussion is given. The OPE of spin three primary field are calculated, in which a primary field with spin four is emerging.

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1.Introduction

The discovery of the AdS_{d+1}/CFT_d correspondence [24, 18, 32] brings the role of conformal field theory to a special stage. The type IIB string theory on the $AdS_5 \otimes S^5$ are equivalent to $\mathcal{N} = 4$ super Yang-Mills in Minkowski space-time. However, the calculation of the correlation functions of physical quantities are limited by our knowledge, except for the CFT_2, saying 2d conformal field theory (CFT) case. With the help of infinitely dimensional symmetries of 2d CFT, much more information can be obtained. For example, Seiberg et al. considered the duality between AdS_3 and CFT_2 [16, 23]. As a special case of this duality, a type IIB string

theory on $AdS_3 \otimes S^3 \otimes T^4$ is equivalent to a certain 2d superconformal field theory (SCFT), which corresponds to the IR limit of the dynamics of parallel $D1$-branes and $D5$-branes. Light-cone gauge quantization of string theories on AdS_3 are given in [34].

For 2d CFT, when the central charge of the theory is greater than one, the Virasoro symmetry must be enlarged [35], or more primary fields should be added, this extended structure is called W symmetry. If there are a finite number of primary fields, then the values of central charge c and conformal weight (or spin) h take on rational values. there are called rational conformal field theories (RCFT), see [4, 12, 20, 31] for review.

The Z_k parafermion (PF) algebra is proposed by Zamolodchikov and Fateev [36] for describing a two-dimensional statistical system with Z_k symmetry associating "spin" variables σ_r to each node r in a (square) lattice L, the σ_r take the N values ω^q ($q = 0,\ 1\ ,\ldots, k-1$), where $\omega = exp(2i\pi/k)$. This generalizes the fermion of the Ising model, which corresponds to the node of Z_2. It is also known that there are various of statistical models, which can be described by this extended theory, such as the 3-state Potts model ($k = 3$) [11], Ashkin-Teller model ($k = 4$) [33]. The Z_4 parafermion also gives a consistent 6d string theory [1].

In fact, parafermion field is important in fractional superstring theory [6], W-string theory [31, 20], furthermore in the compactification of a type II string theories on a Calabi-Yau (CY) manifolds [13, 14] and the construction of $\mathcal{N} = 2$ SCFT [9, 15]. Gepner model is the tensor products of $\mathcal{N} = 2$ minimal models with the internal central charge $c = 9$, which is exactly a solvable models for strings compactified on a CY manifolds. And its applications in Dp-brane theory are also presented in [28, 19] recently. For example, $D0$-branes, the wrapping of Dp-branes on p-dimensional supersymmetric cycles leads to BPS saturated. Its dynamics can be analyzed by a Ishibashi boundary states [21]. The Ishibashi boundary state is the RCFT extension of a boundary state of open sting theories. In open sting theories, the boundary must be chosen such that the 2d CFT symmetry is not broken [5, 17, 27].

$$(L_n - \bar{L}_{-n})|B >= 0, \tag{1.1}$$

here $|B >$ is a boundary state. When the extension structure of the CFT forms a RCFT, the RCFT symmetry on the boundary must be hold also. So that the bulk left- and right-moving primary currents W, $\bar{W}$ have to satisfy certain relations on the boundary. On the construction of the boundary state, the Ishibashi states $|i >>$ have the following relation,

$$(W_n - (-1)^{h_W} \bar{W}_{-n})|i >>= 0, \tag{1.2}$$

where the h_W is the conformal dimension of W. So the explicit expression of W algebra is important and helpful for solving this problem. For the intrinsic relations between the

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Gepner model and the parafermion [14], the construction of W algebra without introducing the free boson has his own advantage.

It is well known that conformal algebra may be obtained from current algebra via Sugawara construction. Similar ways of constructing W-algebra from current algebras were found by Bais et al. [2], through simple third order Casimir in level one case and GKO coset model in $SU(N)_1 \otimes SU(N)_k/SU(N)_{k+1}$ case. The construction of W-algebra directly (not by free field realization) from $SU(2)_k$ parafermion was also proposed [30], in which the so called Z-algebra technique was used. In a sense of that the generating PFs can be defined through the current algebras by projecting out the Cartan subalgebraic valued components, the Z-algebra construction may have the most similarity to the Sugawara construction. On the other hand, parafermion is a coset valued field. Thus the parafermion realization of W_k algebra for specific level may unify the Sugawara and the coset construction. It has been conjectured that all RCFT can be represented as cosets, and that any CFT can be arbitrary well approximated by a rational theory. So the studying of rational theory has his own interesting. As we know, the bosonization representation of a conformal model provide a much bigger Fock space[9, 26, 29]. We have to use the BRST operator ($Q^2_{BRST} = 0$ or other restrictions) to project it onto the physical space. Hence there are much more complications if we use the free field realization. Therefore the advantage of our approach is that it avoids the ambiguity and complexity of the bosonization. It is also worthy to find W_n algebras for PFs of higher rank group. The reason is the follows. The central charges of PFs [13, 15, 36]

$$c = \frac{kD}{k+g} - r, \qquad (1.3)$$

where D, r are dimension, rank of Lie algebra $\mathcal{G}$ respectively, and level k of $\hat{\mathcal{G}}$ is also an integer defining the cyclic symmetry of PFs. In $SU(2)_k$ case $c = 2(k-1)/(k+2)$ agrees with a special case in the Fateev-Lykyanov's W-algebra series $c = (k-1)[1 - k(k+1)/p(p+1)]_{p=k+1}$ [10]. However, there is no known W-algebra, which is constructed from boson, current algebra or coset model, has the central charge coinciding with the PFs construction of groups with higher rank.

In [8, 7, 30] we gave a construction of Virasoro algebra by using non-local fields (parafermions) which take values on coset space $G/U(1)^r$, where G is a simply connected compact Lie group manifold, its Lie algebra $\mathcal{G}$ is a simple one with rank r. There the so called Z-algebra technique was used. We also extended this approach to construct the W-symmetries, W_3 algebra and W_5 algebra were obtained from $SU(2)$ and $SU(3)$, respectively (part results in $SU(2)$ case was reconsidered recently in [25]). While in ref. [8, 7] the construction of W_3 algebra from the $SU(3)$ parafermion was based on a special choice of the root set for summation,

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and turned out that the W algebra were magical closed, while for other choice of root set, the construction was not correct.

As known that in ref. [8, 7, 30] we only obtained the PFs construction of RCFT for $SU(n)(n \leq 3)$ cases, and further extension of this construction was not succeed. In fact, from $SU(4)$ PFs the next and direct goal this construction is failed for spin three primary field. However, it seems that the possibility was not removed at any extent, and the reasons what make these problems arising were unclear at that time. In this paper we will discuss these problems.

The layout of this paper is as follows. In section 2, we recall some basic aspects of extended CFT and the parafermion field, establish our notations and obtain the identities which will be used at a later stage. In section 3, using the relations obtained in section 2, the Virasoro algebra constructed from arbitrary Lie algebra $\mathcal{G}$ PFs is given, the approach presented here greatly simplifies the calculation in [8, 7, 30]. If we hope that the extension structure of RCFT is nontrivial, certain restriction must be put for the Lie algebra root set Φ on which the parafermion fields take values. They coincide with the known results for the $SU(2)$ and $SU(3)$ cases, and get rid of the possibility by this construction from $SU(4)$ PFs for spin three primary field. Assuming the condition of the root set is satisfied, in section 4 we obtain a spin 3 primary field very general in simple-laced case, more detailed discussion is given for A_l algebra case.

2.Brief review of RCFT

In this section we first review some basic aspects of Virasoro algebra, W algebra and parafermion field. Then notations which will be used in the sequel are introduced.

The OPE of the stress momentum tensor is

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \dots, \tag{2.1}$$

the commutator of its modes generate a chiral algebra which is just the Virasoro algebra. To simplify the the expression of OPE, we denote the last equation as,

$$[TT]_4 = c/2, \quad [TT]_3 = 0, \quad [TT]_2 = 2T, \quad [TT]_2 = \partial T, \quad . \tag{2.2}$$

As mentioned previously in the introduction, in the case of the central charge greater than one, it is necessary to enlarge the symmetry of the CFT by adding primary field with spin

great than two [35]. The first nontrivial chiral primary field $W(z)$ of conformal dimension 3 (For the left chiral field its conformal dimension is identical with its spin, so we use them without difference.) with the OPE,

$$W(z)W(w) = \frac{c/3}{(z-w)^6} + \frac{2T(w)}{(z-w)^4} + \frac{\partial T(w)}{(z-w)^3}$$
$$+ \frac{1}{(z-w)^2}(2b^2\Lambda(w) + \frac{3}{10}\partial^2 T)$$
$$+ \frac{1}{z-w}(b^2\partial\Lambda(w) + \frac{1}{15}\partial^3 T) + \ldots, \tag{2.3}$$

$$T(z)W(w) = \frac{3W(w)}{(z-w)^2} + \frac{\partial W(w)}{z-w} + \ldots, \tag{2.4}$$

where $\Lambda(z) = [TT]_0(z) - \frac{3}{10}\partial^2 T(z)$, and constant $b^2 = 16/(22 + 5c)$. The primary feature of W field is governed by the equation (2.4). Identically, we express the above equation as

$$[WW]_6 = c/3, \quad [WW]_5 = 0, \quad [WW]_4 = 2T, \quad [WW]_3 = \partial T,$$

$$[WW]_2 = (2b^2\Lambda(w) + \frac{3}{10}\partial^2 T), \quad [WW]_1 = (b^2\partial\Lambda(w) + \frac{1}{15}\partial^3 T), \tag{2.5}$$

$$[TW]_{6,5,4,3} = 0, \quad [TW]_2 = 3W, \quad [TW]_1 = \partial W. \tag{2.6}$$

Parafermionic currents are primary fields of the 2d CFT. The general parafermion defined for root lattices are proposed in [13]. For a semi-simple Lie algebra $\mathcal{G}$ there are $D - r$ generating parafermion operators ψ_α, where $D = dim\mathcal{G}$ and $r = rank\mathcal{G}$ are dimension and rank of $\mathcal{G}$, respectively. α is a root of $\mathcal{G}$ (analogy to their counter part in the antiholomorphic sector $\bar{\psi}_\alpha$, which will be left out for simplicity). For general parafermion we denote them by a vector in the root lattices M mod a lattice kML, where ML is the long root lattices and k is a constant identified with the level in the corresponding affine Lie algebra $\hat{\mathcal{G}}$ [22]. The generating parafermions are defined through their relationship with current algebra. Thus define the fields

$$\chi_\alpha(z) = \sqrt{\frac{2k}{\alpha^2}} : \psi_\alpha(z)exp(i\alpha\phi(z)/\sqrt{k}) :,$$
$$h_j(z) \equiv h_{\alpha_j}(z) = \frac{2i\sqrt{k}}{\alpha_j^2}\alpha_j\partial_z\phi(z), \tag{2.7}$$

for any root α and simple root α_j. We require that the currents $\chi(z)$ and $h(z)$ (Cartan subgroup valued components) obey the OPE of the current algebra. Because of the mutually

semi-local property between the two parafermions, the radial ordering product is a multival-ued functions, so we can define the radial order product of (generating) parafermions (PFs) $\psi_\alpha(z)$, $\psi_\beta(w)$ (α, β are roots of the underlying Lie group)[8]

$$
\begin{aligned}
&R(\psi_\alpha(z)\psi_\beta(w)) \\
&= \begin{cases} \psi_\alpha(z)\psi_\beta(w), \ \ldots, \omega^{(k-1)\alpha\beta}\psi_\alpha(z)\psi_\beta(w), \ |z| > |w| \, ; \\ \omega^{\alpha\beta/2}\psi_\beta(w)\psi_\alpha(z), \ \ldots \omega^{(k-1/2)\alpha\beta}\psi_\beta(w)\psi_\alpha(z), \ |z| < |w| \, , \end{cases}
\end{aligned} \tag{2.8}
$$

where $\omega = exp(2\pi i/k)$. The RHS of (2.8) is the requirement of analysis for the field. By using (2.8) one therefore have the following relation for the parafermion fields

$$
R\left(\psi_\alpha(z)\psi_\beta(w)\right)(z-w)^{\alpha\beta/k} = R\left(\psi_\beta(w)\psi_\alpha(z)\right)(w-z)^{\alpha\beta/k}. \tag{2.9}
$$

which is an extension of that for fermion (i.e. $\alpha \cdot \beta = 1$, $k = 2$), and boson (i.e. $k \to \infty$). We will drop the R symbol in the following without confusion. The OPE of the parafermion fields defined by[8]

$$
\begin{aligned}
\psi_\alpha(z)\psi_\beta(w)(z-w)^{\alpha\beta/k} \ &= \ \frac{\delta_{\alpha,-\beta}}{(z-w)^2} + \frac{\varepsilon_{\alpha,\beta}/\sqrt{k}}{z-w}\psi_{\alpha+\beta}(w) \\
&\quad + \sum_{n=0}^\infty (z-w)^n[\psi_\alpha\psi_\beta]_{-n}, \\
&\equiv \sum_{n=-2}^\infty (z-w)^n[\psi_\alpha\psi_\beta]_{-n}, \tag{2.10}
\end{aligned}
$$

Which means that we have

$$
[\psi_\alpha\psi_\beta]_l = 0, \ (l \geq 3) \ \ [\psi_\alpha\psi_\beta]_2 = \delta_{\alpha,-\beta}, \ \ [\psi_\alpha\psi_\beta]_1 = \frac{\varepsilon_{\alpha,\beta}}{\sqrt{k}}\psi_{\alpha+\beta} \tag{2.11}
$$

where $\varepsilon_{\alpha,\beta}$ is the structure constant of Lie algebra $\mathcal{G}$ (see more details in the Appendix).

For every field in the parafermion theory there is a pair of charges $(\lambda, \bar{\lambda})$, which take values in the weight lattice. So we denote such field by $\phi_{\lambda,\bar{\lambda}}(z, \bar{z})$ [36, 13, 8]. The OPE of the ψ_α and $\phi_{\lambda,\bar{\lambda}}(z, \bar{z})$ is given by

$$
\psi_\alpha(z)\phi_{\lambda,\bar{\lambda}}(w, \bar{w}) = \sum_{-\infty}^\infty (z-w)^{-m-1-\alpha\lambda}A_m^{\alpha,\lambda}\phi_{\lambda,\bar{\lambda}}(w, \bar{w}) \tag{2.12}
$$

which means that we define the action of the operator (mode) $A_m^{\alpha,\lambda}$ on $\phi_{\lambda,\bar{\lambda}}(z)$ by the integration

$$A_m^{\alpha,\lambda}\phi_{\lambda,\bar{\lambda}}(w,\bar{w}) = \oint_{c_w} dz \, (z-w)^{m+\alpha\lambda}\psi_\alpha(z)\phi_{\lambda,\bar{\lambda}}(w,\bar{w}) \qquad (2.13)$$

where c_w is the contour around w, and for simplicity the notation $\oint dz \equiv \oint \frac{dz}{2\pi i}$ is implied.

Assuming that fields A_α and B_β are arbitrary function of parafermions with parafermion charges α and β. The fields are local (α,or $\beta = 0$) or semilocal ($\alpha = \beta$=root of the underlying Lie algebra). The OPE of them can be written as

$$R(A_\alpha(z)B_\beta(w))(z-w)^{\alpha\beta/k} = \sum_{n=-[h_A+h_B]}^{\infty} [AB]_{-n}(w)(z-w)^n, \qquad (2.14)$$

in which $[h_A]$ means the integral part of dimension A. Hence we have

$$[A_\alpha(z)B_\beta]_n(w)) = \oint_w dz \, A_\alpha(z)B_\beta(w)(z-w)^{n-1+\alpha\beta/k} \qquad (2.15)$$

and some relations

$$[\partial A_\alpha(z)B_\beta]_{-n}(w) = (n+1-\alpha\beta/k)[A_\alpha(z)B_\beta]_{-(n+1)}(w) \qquad (2.16)$$

$$[A_\alpha\partial B_\beta]_n(w) = (n-1+\alpha\beta/k)[A_\alpha(z)B_\beta]_{n-1}(w) + \partial[A_\alpha(z)B_\beta]_n(w) \qquad (2.17)$$

$$[\partial A_\alpha(z)B_\beta]_{-n}(w) + [A_\alpha\partial B_\beta]_{-n}(w) = \partial[A_\alpha(z)B_\beta]_{-n}(w) \qquad (2.18)$$

$$\begin{aligned}
[\partial^n A_\alpha(z)B_\beta]_0(w) &= (n-\alpha\beta/k)\ldots(1-\alpha\beta/k)[A_\alpha(z)B_\beta]_{-n}(w) \\
&\equiv \frac{\Gamma(n+1-\alpha\beta/k)}{\Gamma(1-\alpha\beta/k)}[A_\alpha(z)B_\beta]_{-n}(w) \qquad (2.19)
\end{aligned}$$

in which the Γ is the usual Γ- unction. It is easy to find a relation between three-fold radial ordering products

$$\begin{aligned}
\Big\{ &\oint_w du \oint_w dz \, R(A(u)R(B(z)C(w))) \\
&- \oint_w dz \oint_w du \, (-)^{\alpha\beta/k}R(B(z)R(A(u)C(w))) \\
&- \oint_w dz \oint_z du \, R(R(A(u)B(z))C(w)) \Big\} \\
&(z-w)^{p-1+\beta\gamma/k}(u-w)^{q-1+\gamma\alpha/k}(u-z)^{r-1+\alpha\beta/k} = 0, \qquad (2.20)
\end{aligned}$$

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where the integers p, q, r are in the region $-\infty < p \leq [h_B + h_C]$, $-\infty < q \leq [h_C + h_A]$, $-\infty < r \leq [h_A + h_B]$, and α, β, γ are parafermionic charges of the fields A, B, and C respectively. This equation is an extension of the identity $A(BC) - B(AC) - [A, B]C = 0$. The contours are self evident. Performing the binomial expansion, we can rewrite the last equation as

$$
\begin{aligned}
\oint_w du \oint_w dz \; & R(A(u)R(B(z)C(w))) \sum_{i=p}^{\infty} C_{r-1+\alpha\beta/k}^{(i-p)} \\
& \times (z-w)^{i-1+\beta\gamma/k}(u-w)^{Q-1+(\beta+\gamma)\alpha/k} \\
+ (-1)^r \oint_w dz \oint_w du \; & R(B(u)R(A(z)C(w))) \sum_{j=q}^{\infty} C_{r-1+\alpha\beta/k}^{(j-q)} \\
& \times (z-w)^{Q-j-1+\beta(\alpha+\gamma)/k}(u-w)^{j-1+\gamma\alpha/k} \\
= \oint_w dz \oint_z du \; & R(R(A(u)B(z))C(w)) \sum_{l=r}^{\infty} C_{q-1+\alpha\gamma/k}^{(l-r)} \\
& \times (z-w)^{Q-l+(\alpha+\beta)\gamma/k}(u-z)^{l-r+\beta\alpha/k},
\end{aligned}
\tag{2.21}
$$

From the two equations above we obtain the following Jacobi-like identity relations[8, 30]

$$
\begin{aligned}
\sum_{i=p}^{[h_B+h_C]} & C_{r-1+\alpha\beta/k}^{(i-p)}[A[BC]_i]_{Q-i}(w) \\
+ (-)^r \sum_{j=q}^{[h_C+h_A]} & C_{r-1+\alpha\beta/k}^{(j-q)}[B[AC]_j]_{Q-j}(w) \\
= \sum_{k=r}^{[h_B+h_A]} & (-)^{(k-r)} C_{q-1+\alpha\gamma/k}^{(k-r)}[[AB]_k C]_{Q-k}(w),
\end{aligned}
\tag{2.22}
$$

in which $Q = p+q+r-1$, $C_x^{(l)} = \frac{(-)^l x(x-1)\ldots(x-l+1)}{l!}$, and $C_0^{(0)} = C_n^{(0)} = C_{-1}^{(l)} = 1$, $C_p^{(l)} = 0$, for $p, l > 0, l > p$. This identity is important for our usage, we will use it extensively. Performing analytic continuation one more equation is obtained

$$
[BA]_r(w) = \sum_{i=r}^{[h_A+h_B]} \frac{(-)^t}{(t-r)!} \partial^{t-r}[AB]_t(w),
\tag{2.23}
$$

and two special cases should be mentioned $(n \geq 0)$

$$
[Aconst.]_{-n}(w) = const.\frac{1}{n!}\partial^n A(w), \quad [const.A]_{-n}(w) = const.\delta_{n,0}B(w).
\tag{2.24}
$$

In all of the previous equations A, B, C can be compound operators. We can calculate any coefficient in OPE from fundamental equation (2.10).

3.PFs constructions CFT

In this section, we present an another approach beside the Z-algebra technique used in[8]. This approach greatly simplifies the calculation of [8], and the restriction on the root set arises naturally from the definition of the primary parafermion field. The detailed derivation of OPE of the stress momentum tensor is given.

For the notation conveniences, define $N_{\alpha,\beta} \equiv \varepsilon_{\alpha,\beta}/\sqrt{k}$, and the following identities are hold by $N_{\alpha,\beta}$,

$$N_{\alpha,\beta} = -N_{\beta,\alpha} = -N_{-\alpha,-\beta} = \frac{(\alpha+\beta)^2}{\beta^2} N_{-\alpha,\alpha+\beta}. \tag{3.1}$$

If we only consider the simple-laced case, the results are

$$N_{\alpha,\beta} = -N_{\beta,\alpha} = -N_{-\alpha,-\beta} = N_{-\alpha,\alpha+\beta}. \tag{3.2}$$

Further more, we define $\tau_\alpha = [\psi_\alpha\psi_{-\alpha}]_0$, $\eta_\alpha = [\psi_\alpha\psi_{-\alpha}]_{-1}$, $\Omega_\alpha = [\psi_\alpha\psi_{-\alpha}]_{-2}$. We calculate the OPE of τ_α with ψ_α, and τ_β. The results coincide with the OPE of stress momentum tensor, and the modes of τ_β give the Virasoro algebras.

From the definition of τ_β, and the (2.22), obviously we have,

$$\tau_\alpha = \tau_{-\alpha}, \tag{3.3}$$

$$[\tau_\alpha\psi_\beta]_l = 0, (l \geq 3), \tag{3.4}$$

setting $Q = p = 2$, $q = 1$, $r = 0$ in the (2.22), we have

$$[\tau_\alpha\psi_\beta]_2 \tag{3.5}$$

$$= [\tau_{-\alpha}\psi_\beta]_2$$

$$= [\psi_\alpha[\psi_{-\alpha}\psi_\beta]_2]_0 + [\psi_{-\alpha}[\psi_\alpha\psi_\beta]_1]_1 + (1 + \alpha^2/k)[\psi_{-\alpha}[\psi_\alpha\psi_\beta]_2]_0$$

$$- \frac{\alpha\beta}{k}[[\psi_\alpha\psi_{-\alpha}]_1\psi_\beta]_1 + \frac{\alpha\beta}{2k}\left(1 - \frac{\alpha\beta}{k}\right)[[\psi_\alpha\psi_{-\alpha}]_2\psi_\beta]_0$$

$$= \delta_{\alpha,\beta}\psi_\alpha + N_{\alpha,\beta}N_{-\alpha,\alpha+\beta}\psi_\beta + (1 + \alpha^2/k)\delta_{\alpha,\beta}\psi_{-\alpha}$$

$$+ \frac{\alpha\beta}{2k}(1 - \frac{\alpha\beta}{k})\psi_\beta$$

$$= \delta_{-\alpha,\beta}\psi_{-\alpha} + N_{-\alpha,\beta}N_{\alpha,-\alpha+\beta}\psi_\beta$$

$$+ (1 + \alpha^2/k)\delta_{-\alpha,\beta}\psi_\alpha - \frac{\alpha\beta}{2k}(1 + \frac{\alpha\beta}{k})\psi_\beta. \tag{3.6}$$

We denote $\tau = \sum_{\alpha \in \Phi} \tau_\alpha$, where the Φ is the root set for summation. We require that the set satisfy the conditions

$$\{\Phi\} \bigcap \{-\Phi\} = \emptyset, \ \{\Phi\} \bigcup \{-\Phi\} = \Delta,$$

Obviously the number of roots in Φ equals the number of ones in P. In fact, the Φ can be obtained from P by some Weyl reflections. From the equation (3.6) we obtain

$$[\tau \psi_\beta]_2 = \left(1 + \sum_{\alpha \in \Phi} (\alpha\beta/2k - (\alpha\beta)^2/2k^2 + N_{\alpha,\beta} N_{-\alpha,\alpha+\beta}) \right) \psi_\beta, \ (\beta \in \Phi), \tag{3.7}$$

where without loosing generality we choose $\beta \in \Phi$ for convenience, and we will not mention it in the later stage. From the general theory of the conformal fields [3, 35], we know that the conformal dimension of the parafermion ψ_α is $(1 - \alpha^2/2k)$. We normalize the τ to

$$T = \frac{k}{k+g}\tau, \tag{3.8}$$

in which the g is the dual Coxeter number, and k is the level of the representation of $\hat{\mathcal{G}}$, which is the affinization of the classical Lie algebra $\mathcal{G}$. For a consistent theory, we require

$$[T\psi_\beta]_2 = \left(1 - \frac{\beta^2}{2k} \right) \psi_\beta \tag{3.9}$$

or, equivalently,

$$[\tau\psi_\beta]_2 = \left(1 + \frac{2g - \beta^2}{2k} - \frac{g\beta^2}{2k^2} \right) \psi_\beta \tag{3.10}$$

Comparing the last equation with (3.7), we get the following conditions for set Φ.

$$\sum_{\alpha \in \Phi} \left(\frac{\alpha\beta}{2k} + N_{\alpha,\beta} N_{-\alpha,\alpha+\beta} \right) = \frac{2g - \beta^2}{2k}, \tag{3.11}$$

$$\sum_{\alpha \in \Phi} (\alpha\beta)^2 = \sum_{\alpha \in P} (\alpha\beta)^2 = g\beta^2. \tag{3.12}$$

The last two equations are just the consistent condition for PFs construction of CFT. From the definition of g we know that the condition (3.12) is satisfied for any given Lie algebra ($\psi^2 = 2$). While the condition (3.11) brings a constraint on root system of $\mathcal{G}$. Therefore we get a necessary condition for the root set on which the summation is defined for a consistent theory. From (3.6) we obtain:

$$k \sum_{\alpha \in \Phi} (N_{\alpha,\beta} N_{-\alpha,\alpha+\beta} - N_{-\alpha,\beta} N_{\alpha,-\alpha+\beta}) = \beta^2, \tag{3.13}$$

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while on the other hand, we have

$$k \sum_{\alpha \in \Phi} (N_{\alpha,\beta} N_{-\alpha,\alpha+\beta} + N_{-\alpha,\beta} N_{\alpha,-\alpha+\beta}) = 2g - 2\beta^2. \tag{3.14}$$

So we get the solution

$$k \sum_{\alpha \in \Phi} N_{\alpha,\beta} N_{-\alpha,\alpha+\beta} = \frac{2g - \beta^2}{2}, \tag{3.15}$$

$$k \sum_{\alpha \in \Phi} N_{\alpha,-\beta} N_{-\alpha,\alpha-\beta} = \frac{2g - 3\beta^2}{2}, \tag{3.16}$$

and we have (if $N_{\alpha,\beta} \neq 0$, or, $N_{\alpha,-\beta} \neq 0$)

$$\sum_{\alpha \in \Phi} (\alpha\beta) = 0, \tag{3.17}$$

in which β is an arbitrary element of the Δ, so we can re-express the last identity as,

$$\sum_{\alpha \in \Phi} \alpha = 0, \tag{3.18}$$

This is the condition for root system $\mathcal{G}$ on which the summation will be taken over. Which says that for very simple root α_i the sum of his height in Φ must be zero. For $SU(3)_k$ as an example $\Phi = \{\alpha_1, \ \alpha_2, \ \alpha_3 = -(\alpha_1 + \alpha_2,)\}$, this coincides with the result in [8]. In this paper we only consider the simple-laced cases for simplicity, saying $\alpha^2 = 2$, and $\alpha\beta = -1$, if $\alpha + \beta \in \Delta$. When $g = 2$, $N_{\alpha,\beta} = 0$, this is a special case, and we have $\alpha = \beta$, this is the $SU(2)_k$ (Z_k symmetry). please see [30] for more details. While for $g \geq 2$, we have the following identities:

$$k \sum_{\alpha \in \Phi} N_{\alpha,\beta} N_{-\alpha,\alpha+\beta} = g - 1, \quad k \sum_{\alpha \in \Phi} N_{-\alpha,\beta} N_{\alpha,-\alpha+\beta} = g - 3, \tag{3.19}$$

$$k \sum_{\alpha \in \Phi} \alpha\beta N_{\alpha,\beta} N_{-\alpha,\alpha+\beta} = -(g - 1), \quad k \sum_{\alpha \in \Phi} \alpha\beta N_{-\alpha,\beta} N_{\alpha,-\alpha+\beta} = g - 3, \tag{3.20}$$

$$\sum_{\alpha \in \Phi} (\alpha\beta)^2 = 2g, \quad \sum_{\alpha \in \Phi} (\alpha\beta)^3 = 6, \quad \sum_{\alpha \in \Phi} (\alpha\beta)^4 = 2g + 12. \tag{3.21}$$

The proof of the above identities is very simple. Using these identity (we will not mention them separately), we have

$$[T\psi_\beta]_2 = (1 - 1/k)\psi_\beta \tag{3.22}$$

repeating the same procedure, we have

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$$[T\psi_\beta]_1 = \partial\psi_\beta \tag{3.23}$$

in the process of deriving the last equation, the identity,

$$\sum_{\alpha\in\Phi} (N_{\alpha,\beta}[\psi_{-\alpha}\psi_{\alpha+\beta}]_0 + N_{-\alpha,\beta}[\psi_\alpha\psi_{-\alpha+\beta}]_0)$$
$$= \frac{1}{2} \sum_{\alpha\in\Phi} (N_{\alpha,\beta}N_{-\alpha,\alpha+\beta} + N_{-\alpha,\beta}N_{\alpha,-\alpha+\beta}) \partial\psi_\beta, \tag{3.24}$$

is used. We can express the results as the OPE

$$T(z)\psi_\beta(w) = \frac{1 - 1/k}{(z-w)^2} + \frac{1}{z-w}\partial\psi_\beta(w) + \dots. \tag{3.25}$$

Repeating the same process for T, one can get the OPE of the $T(z)T(w)$. We leave out the detail of the all, and just give one example of them, saying

$$
\begin{aligned}
[\tau_\alpha\tau_\beta]_2 \quad &= [[\tau_\alpha\psi_\beta]_1\psi_{-\beta}]_1 + [[\tau_\alpha\psi_\beta]_2\psi_{-\beta}]_0 + [\psi_\beta[\tau_\alpha\psi_{-\beta}]_2]_0 \\
&= \delta_{\alpha,\beta}[\psi_\alpha\psi_\beta]_0 + (1 + \alpha^2/k)\delta_{-\alpha,-\beta}[\psi_\beta\psi_{-\alpha}]_0 \\
&\quad + \delta_{\alpha,\beta}[\psi_\beta\psi_\alpha]_0 + (1 + \alpha^2/k)\delta_{-\alpha,\beta}[\psi_{-\alpha}\psi_\beta]_0 \\
&\quad + (N_{\alpha,\beta}N_{-\alpha,\alpha+\beta} + N_{-\alpha,\beta}N_{\alpha,-\alpha+\beta} - \frac{(\alpha\cdot\beta)^2}{k^2})\tau_\beta \\
&\quad + N_{\alpha,\beta}[[\psi_{-\alpha}\psi_{\alpha+\beta}]_0\psi_{-\beta}]_1 + N_{-\alpha,\beta}[[\psi_\alpha\psi_{-\alpha+\beta}]_0\psi_{-\beta}]_1 \\
&\quad + \frac{\alpha\beta}{k}(1 + \alpha^2/k)(\delta[\psi_\alpha\psi_{-\beta}]_0 - \delta_{-\alpha,\beta}[\psi_{-\alpha}\psi_{-\beta}]_0),
\end{aligned} \tag{3.26}
$$

consider the summation for α and β, we have

$$[TT]_2 = 2T, \tag{3.27}$$

the other terms can be derived in the same manner, they read

$$[TT]_4 = c/2, \quad [TT]_3 = 0, \quad [TT]_1 = \partial T, \tag{3.28}$$

equivalently, we can re-express these results as

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \dots, \tag{3.29}$$

in which the central charge c is given by formula (1.3). The last equation is the OPE of the stress momentum tensor. In fact, notice that $\tau_\alpha = \tau_{-\alpha}$, and there is only single index α needs

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for summation, we can extend the summation over Φ to Δ without difficulty for τ_α. And if we choose the root set Δ for summation, no essential differences will arise, for $\sum_{\alpha \in \Delta} \alpha = 0$. The only difference is the normal constant, which are two times of the original one. So the PFs construction for Virasoro can be extended to any given Lie algebras. Therefore we obtain parafermion representation of the Virasoro algebras underlying any given Lie algebra with arbitrary level k. It is trivial to extend this construction to semi-simple Lie algebra case, if the condition is satisfied by every copy of the simple Lie algebra.

4.PFs realization of RCFT with spin 3

In this section we will construct a spin three field, calculate the OPE of the field with T. It turns out that the field is a primary field. This is one of the main features of W_3 algebra. We conjecture that this field is the first primary field in W algebras. We then calculate the OPE of the spin 3 field with itself, and in which a spin 4 primary field emerges.

Using the notation introduced in the previous, i.e. $\eta_\beta \equiv [\psi_\beta \psi_{-\beta}]_{-1}$, further we define $w_\beta \equiv \eta_\beta - \eta_{-\beta}$, and $w_3 = \sum_{\beta \in \Phi} w_\beta$. Obviously, $w_\beta = -w_{-\beta}$, $\sum_{\beta \in \Phi} w_\beta = -\sum_{\beta \in -\Phi} w_\beta$.

Perform the same calculation by properly choosing of Q, p, q in the Jacobi-like identity (the final result is the same for different choice, but for certain choice the calculation becomes simpler), and we have,

$$[Tw_3]_5 = [Tw_3]_4 = [Tw_3]_3 = 0, \tag{4.1}$$

while for example, setting $Q = r = 1$, $q = 2$, we have

$$\begin{aligned} [T\eta_\beta]_2 &= [\psi_\beta [T\psi_{-\beta}]_2]_{-1} + [[T\psi_\beta]_1 \psi_{-\beta}]_0 + [[T\psi_\beta]_2 \psi_\beta]_{-1} \\ &= (2 - 2/k)\eta_\beta + [\partial \psi_\beta \psi_{-\beta}]_0 \\ &= 3\eta_\beta, \end{aligned} \tag{4.2}$$

therefore we have,

$$[Tw_\alpha]_2 = 3w_\alpha, \quad or, \quad [Tw_3]_2 = 3w_3. \tag{4.3}$$

In the process for deriving those results, some known identities are used without mention. Similarly, we obtain,

$$[Tw_3]_1 = \partial w_3. \tag{4.4}$$

Equivalently, the OPE expression is

$$T(z)w_3(w) = \frac{3w_3(w)}{(z-w)^3} + \frac{\partial w_3(w)}{z-w} + \dots \tag{4.5}$$

and we complete the proof for the w_3 that it is a spin three primary field.

However, from $\sum_{\beta \in \Phi} w_\beta = -\sum_{\beta \in -\Phi} w_\beta$, we know that the summation of w_α defined on the root set Δ is identical to zero. It means that we cannot find any extension of the Virasoro algebras (W-algebra) on the total root system Δ for any Lie algebras $\mathcal{G}$. If we expect that such extension to be existence, one part of the roots (we denote it by Φ), on which the spin three field are defined, must be separated out. While the restriction of the central charge require that the number of the elements in Φ is the half of the ones in Δ, or is the same as in P. On the other hand, the consistence of the theory brings more constraints on the roots set Φ. We can express them as

Proposition: The parafermion representation of spin three primary field w_3 is invariant under the Weyl reflection up to a minus one.

$$s(w_3) = \pm w_3, \tag{4.6}$$

$$s_\beta(w_\alpha) = w_\alpha - \frac{2(\alpha\beta)}{\beta^2} w_\beta, \quad \alpha, \ \beta \in \Phi. \tag{4.7}$$

Starting from this, we have the following relation for the height of Φ, it reads,

$$h_\Phi = h_\Phi \pm h_\Phi, \tag{4.8}$$

then the solution of the equation is $h_\Phi = 0$, which recover the consistent condition. We used the Z-algebra technique to construct W_3-algebra for $SU(3)$ PFs in[8], for simplicity we choose the symmetric roots $\Phi = \{\alpha_1, \alpha_2, \alpha_3 = -(\alpha_1 + \alpha_2)\}$ for summation there. At now we see that, this choice is essential and unique. Here, we have no enough space to list out the calculation in detail. The OPE of $W_3(z)W_3(w)$ $(g > 2)$ are

$$\begin{aligned}
W_3(z)W_3(w) \quad &= \frac{c/3}{(z-w)^6} + \frac{2T}{(z-w)^4} + \frac{\partial T}{(z-w)^3} \\
&\quad + \frac{1}{(z-w)^2}\left(2b^2\Lambda(w) + \frac{3}{10}\partial^2 T(w) + V(w)\right) \\
&\quad + \frac{1}{z-w}\left(b^2\partial\Lambda(w) + \frac{1}{15}\partial^3 T(w) + \partial V(w)\right), \tag{4.9}
\end{aligned}$$

$$W_3 \quad = \left(\frac{k^3}{6(k-2)(k+1)(k+g)}\right)^{1/2} w_3, \tag{4.10}$$

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in which

$$V(z) = \frac{2(4k+3)k^2}{3(k-2)(k+1)(k+g)} \sum_{\alpha \in \Phi} (\Omega_\alpha + \Omega_{-\alpha})$$
$$- \frac{2k^2}{(k-2)(k+1)(k+g)} \sum_{\alpha,\beta \in \Phi} \alpha\beta[\tau_\alpha\tau_\beta]_0 - 2b^2[TT]_0$$
$$+ \left(\frac{3}{10}(2b^2-1) - \frac{k}{2(k-2)}\right)\partial^2 T. \tag{4.11}$$

is a spin four primary field, which can be proven from general CFT [3], or by directly calculation. It is very obviously that $\sum_{\alpha,\beta \in \Delta} \alpha\beta[\tau_\alpha\tau_\beta]_0 = 0$, and the left parts of V is not a primary field anymore. So the definition of V cannot be extended to the root set Δ, For $g = 3$, saying the $SU(3)_k$ case, its expression reduce to[8]

$$V(z) = \frac{2(4k+3)k^2}{3(k-2)(k+1)(k+3)} \sum_{\alpha \in \Phi} (\Omega_\alpha + \Omega_{-\alpha})$$
$$+ \frac{2k^2}{(k-2)(k+1)(k+3)} \sum_{\alpha \in \Phi} [\tau_\alpha\tau_\alpha]_0$$
$$+ \left(\frac{2(k+3)}{3(k-2)(k+1)} - 2b^2\right)[TT]_0$$
$$+ \left(\frac{3}{10}(2b^2-1) - \frac{k}{2(k-2)}\right)\partial^2 T. \tag{4.12}$$

which is null at $k = 3$ [8], so the algebra is closed. The detailed calculation can be fund [7]. However, for $g \geq 4$ cases, the full solution of this problem is still open for their complexity. In the scene that a spin 4 primary field emerges, the algebra is not closed. However, for higher rank Lie algebra, the central charge, which reflects the character of symmetry is larger. From the general theory of CFT[3, 35], we know that the more independent primary fields are needed for the larger central charge. From this point view, it is natural that the field is not closed at spin 3. Unfortunately, we do not find an effective approach to calculate the number of the independent primary fields at now. For the well known method to enumerate the independent generating fields is the so called "character technique" [4]. By that technique, the independent generating fields is the same as the number of independent Casimirs of $\mathcal{G}$. From this point view, no spin three primary field will emerge in $SU(2)$ case, and this is indeed the fact from other approaches. But we have been obtained a W_5 algebra from the $SU(2)$ PFs.

From the above discussion that the root set Φ forms a closed cycle in the root space. However, for a lot of Lie algebras, this condition cannot be satisfied. In fact, we know that

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the set Φ can be obtained from the positive system P by some appropriate Weyl reflection. Obviously, this requires the height of the P to be:

$$h_p = 2 \sum_{\alpha \in P} n_\alpha, \quad h_\rho = \sum_{\alpha \in P} n_\alpha, \tag{4.13}$$

where $n_\alpha \in N$, is the times of the α emerging in P. (4.13) says that, for every simple root α_i, his times appearing in P must be an even number. Obviously, one cannot find such set Φ for many Lie algebras. For A_l algebra, the positive system is $P = \{\alpha_1, \alpha_2, \ldots \alpha_l, \alpha_1 + \alpha_2, \ldots, \alpha_{l-1} + \alpha_l, \alpha_1 + \alpha_2 + \alpha_3, \ldots, \alpha_{l-2} + \alpha_{l-1} + \alpha_l, \ldots \ldots, \alpha_1 + \alpha_2 + \ldots + \alpha_{l-1} + \alpha_l\}$, the last one in it is the highest root. It is obviously that, the height of simple roots α_j in P (the sum of the multiplicities of α_j as an element of a root in P) is $h_{\alpha_j}(A_l) = j(l - j + 1)$. So in general, the set Φ does not exist for any algebras $A_{2n+1}, (n \geq 1)(h_{\alpha_1} = 2n+1, \ h_{\alpha_2} = 4n, ...)$, and Φ exists for algebras A_{2n} $(n \geq 1)$ (The height labeled by arbitrary simple root α_i is an even number. For example, the height $h_{\alpha_1} = 2n$ and $h_{\alpha_2} = 2(2n - 1)$). For $D_1 = A_1$, $D_2 = A_1 \oplus A_1$, so there are no problem in these two cases; while for $D_3 = A_3$, AND NO solution can be found in this case. For simplicity we leave the discussion of the algebras $D_l(\geq 4)$, E_6, E_7, E_8 and non-simple laced algebras to other place.

5. Discussion

In this paper, we consider a construction of W-symmetries (algebras) through a special kind of coset currents, the non-local currents (parafermion), which take values on the coset space $G/U(1)^r$, where rank r Lie group G is limited to a simply connected compact one. It turns out that, the restriction given by the W-symmetry on the PFs underlying Lie algebra is very stringent. Our extension is very general, all of the previously known results of PFs construction are very simple example of the present discussion. In fact, the present discussion can be generalized to semi-simple Lie algebra case, if the condition of the root set is hold for every copy of its simple Lie algebra.

The important property of OPE is that the singular parts of it can be governed by certain algebra. In PFs construction we use part of the regular terms of the parafermionic OPE , and they satisfy certain algebras also. The property and relation of their higher order terms need further studying.

Because of the the definition of RCFT invariant boundary state is directly relevant to W-algebra, and the important role played by boundary state in Dp-brane theory, it is reasonable to expect that W-algebra (geometry) would be found his position in the Dp-brane theory studying. The number of independent primary fields are relevant to fusion rule and character.

So the problems need to further explore.

For ψ_α is primary currents, from the general discussion of the boundary state of open string, it seems that, a kind of new boundary state

$$(\psi_n^\alpha \pm (-1)^{-\alpha^2/2k}\psi_{-n}^\alpha)|B >= 0. \tag{5.1}$$

should exist. By adding appropriately chosen $U(1)$ current, spin 1 currents or spin $3/2$ supercurrents, and their corresponding boundary states can be obtained.

6. Appendix

For the convenience of usage, here we recall some basic data of the root system of the Lie algebras[22].

Any simple Lie algebra can be classified to the classical series A_l, B_l, C_l, D_l, and the except ones, E_6, E_7, E_8, F_4 and G_2. There are at most two different length roots for any of them. For our purposes we choose the basis as

$$[E_\alpha, E_\beta] = \varepsilon_{\alpha,\beta}E_{\alpha+\beta} \tag{6.1}$$

Denote the root set of $\mathcal{G}$ by Δ, We have $\varepsilon_{\alpha,\beta} \neq 0$, if $\alpha + \beta \in \Delta$, $\varepsilon_{\alpha,\beta} = 0$, if $\alpha + \beta \notin \Delta$, and

$$\varepsilon_{\alpha,\beta} = -\varepsilon_{\beta,\alpha} = -\varepsilon_{-\alpha,-\beta} = \frac{(\alpha+\beta)^2}{\beta^2}\varepsilon_{-\alpha,\alpha+\beta}. \tag{6.2}$$

Let $\Pi = \{\alpha_1, \alpha_2, \ldots, \alpha_r\}$ be a simple system of roots of $\mathcal{G}$, r be the rank of $\mathcal{G}$, and Let P be the corresponding positive system, then we have:

(i)$\{P\} \cap \{-P\} = \emptyset$, $\{P\} \cup \{-P\} = \Delta, \alpha$, $\beta \in P$, $\alpha + \beta \in \Delta, \Rightarrow \alpha + \beta \in P$;

(ii)If, $1 \leq i$, $j \leq r$, then $\alpha_i - \beta_j \notin P$, $\alpha_i \cdot \beta_j \leq 0$;

(iii)If $\alpha \in P, \alpha = \sum_i^r m_i\alpha_i$, $m_i \geq 0$, while ρ is half the sum of the P, $\rho = \frac{1}{2}\sum_{\alpha\in P}\alpha$. Then the height of the root α is $h_\alpha = \sum_i^r m_i$; further more we define the total height of the root system P, $h_t = \sum_{\alpha\in P}h_\alpha$, obviously, $h_t = 2h_\rho$;

(iv)Weyl reflection $s_\beta(\alpha) = \alpha - \frac{2(\alpha,\beta)}{(\beta,\beta)}\beta \equiv \alpha - \frac{2\alpha\beta}{\beta^2}\beta$.

If M generate an irreducible representation of $\mathcal{G}$, Let Q_m be the quadratic Casimir operator of the representation, it has an unique highest weight λ, then,

$$Q_m = \lambda(\lambda + 2\rho). \qquad (6.3)$$

For a adjoint representation of the Lie algebra, denoting the accompanying quadratic Casimir operator as Q_ψ, ψ is the highest weight of the adjoint representation. Then we introduce

$$g = Q_\psi/\psi^2 = 1 + 2\rho\psi/\psi^2, \qquad (6.4)$$

using the data given previous, we have

$$\psi/\psi^2 = \sum_{i=1}^{r} m_i \alpha_i/\alpha_i{}^2, \qquad (6.5)$$

and so,

$$g = 1 + \sum_{i=1}^{r} m_i = \sum_{i=0}^{r} m_i, \qquad (6.6)$$

where g is the so-called the dual Coxeter number of the affine Lie algebra[22]. More directly we can use the Freudenthal-de Vries Strange formula:

$$\frac{|\rho|^2}{g} = \frac{Dim\mathcal{G}}{12} \equiv \frac{D}{12}, \qquad (6.7)$$

In the $A - D - E$ cases, $g = 1 + h = 1 + r$, where the h is the height of the highest root.

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